AD-A158 451 SOME SIMPLE MODELS FOR CONTINUOUS VARIATE TIME SERIES (U) NAVAL POSTGRADUATE SCHOOL MONTEREY CA P A LEWIS JUN 85 NPS55-85-009 1/1

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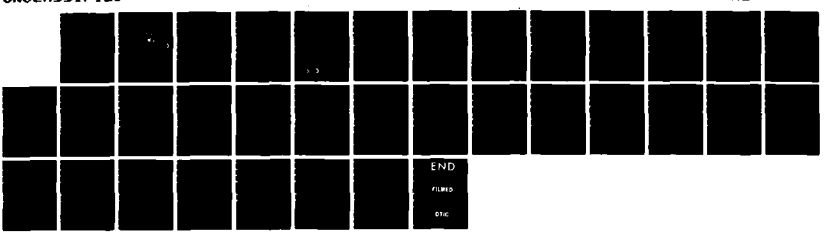

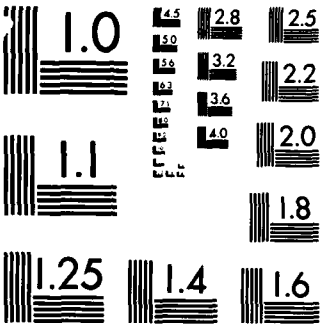

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NPS55-85-009

NAVAL POSTGRADUATE SCHOOL

Monterey, California

SOME SIMPLE MODELS FOR CONTINUOUS VARIATE TIME SERIES

by

P. A. W. Lewis

June 1985

Prepared for:
Office of Naval Research
Arlington, VA 22217

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS55-85-009	2. GOVT ACCESSION NO. A158457	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SOME SIMPLE MODELS FOR CONTINUOUS VARIATE TIME SERIES		5. TYPE OF REPORT & PERIOD COVERED Technical 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) P. A. W. Lewis		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93943-5100		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61153N; RR014-05-01 N0001485WR24062
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, VA 22217		12. REPORT DATE June 1985 13. NUMBER OF PAGES 30
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Continuous variate time series, Exponential, Gamma, Weibull, Laplace, Beta, Residual analysis

mathematics.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A survey is given of recently developed models for continuous variate non-Gaussian time series. The emphasis is on marginally specific models with given correlation structure. Exponential, Gamma, Weibull, Laplace, Beta and Mixed Exponential models are considered for the marginal distributions of the stationary time series. Most of the models are random coefficient, additive linear models. Some discussion of the meaning of autoregression and linearity is given, as well as suggestions for higher-order linear residual analysis for nonGaussian models.

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S/N 0102-LF-014-6601

Unclassified

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Some Simple Models for Continuous Variate Time Series

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ABSTRACT

A survey is given of recently developed models for continuous variate nonGaussian time series. The emphasis is on marginally specific models with given correlation structure. Exponential, Gamma, Weibull, Laplace, Beta and Mixed Exponential models are considered for the marginal distributions of the stationary time series. Most of the models are random coefficient, additive linear models. Some discussion of the meaning of autoregression and linearity is given, as well as suggestions for higher-order linear residual analysis for nonGaussian models.

INTRODUCTION

Most time series analysis of continuous variate discrete time parameter phenomena is based on the simple, linear, autoregressive-moving average processes which were first introduced by Yule at the end of the 19th century and then extended by many authors. Although these processes are not necessarily defined as processes in Gaussian variates, it is simplest to use them as such because linear operations on Gaussian variates preserves Gaussianity. Again, statistical analyses based on the Gaussianity assumption and these models are well developed. Consequently, it has been the practice to either ignore the issue of the marginal distribution of the variates, or to assume that the variates can be transformed mildly to Gaussianity by standard methods.

Unfortunately, there are many areas, particularly in the physical sciences, where nonGaussianity of the time-series is gross and is itself of interest in the modelling of the phenomenon under study. The nonGaussianity may be due to the fact that the phenomena are inherently positive-valued, or are distributed with longer or shorter tails than exhibited by Gaussian variates. The first case is well illustrated in river-flow studies (see e.g., Lawrance and Kottegoda, 1977), as well as in the extensive studies of wind velocity amplitudes by Oceanographers and Meteorologists (see e.g., Brown, Katz and Murphy, 1984). The second case occurs when the North-South and East-West components of wind velocities are studied and it is also important in the study of acoustical phenomena. These acoustical phenomena have marginal distributions which range from almost uniform distributions to very long-tailed distributions which are often modeled as Laplace distributions.

As a consequence, there has been an accelerating effort during the last decade to derive models for stationary nonGaussian continuous variate time series. These models tend to imitate the linear (Gaussian) time series in

their correlation structure and use various standard parametric models for marginal distributions. The present paper will survey some of these models. The emphasis is on models which are (Lewis, 1980) simple and flexible in the following senses:

(i) The models should be specified in terms of easily observed and measured quantifiers. For stationary models the quantifiers should typically be:
    (a) the marginal distribution;
    (b) second-order moments (correlations); and
    (c) simplified higher-order moments or residuals.

(ii) The models should be parametrically parsimonious and parametrically simple.

(iii) The models should be easy to generate on computers, i.e. they should be structurally simple, like the usual linear models.

(iv) The models should be easy to fit to data, both formally and informally.

Note especially point (i.c). The emphasis in deriving the models described below is on the marginal distribution and the correlation structure. However, many models with, say, the same second-order autoregressive structure and Exponential marginal distributions are now known, but the sample path structures of the various models can be vastly different. As a consequence, Lawrance and Lewis (1985a, 1985b) have attempted to introduce a residual analysis for these processes; this residual analysis will be described briefly in the last section.

The work described in this paper is, perforce, mainly that of the author and his colleagues, notably A. J. Lawrance and E. McKenzie; not suprisingly, the work fits the prescription given above. The state of this work can be summarized as follows. Many models with first-order autoregressive structure of some sort and standard parametric marginal distributions

(Exponential, Gamma, Mixed Exponential, Weibull, Laplace, Beta) are now known. Extension to higher-order autoregressive structures is difficult, except in the case of the Exponential marginal distribution, or by using a rather limited random indexing device (mixture device) due to Jacobs and Lewis (1983). Mixed autoregressive-moving average processes can be obtained from most of the models, because they have (random coefficient) linear additive structure. Another problem is, that for positive-valued marginal distributions, it is difficult to get negative correlation in simple ways; this may or may not be a serious problem in, say, hydrology because positive correlation in data seems most common. The problem of negative correlation is intimately tied up with symmetry of the marginal distribution and is, therefore, easy to obtain for the models which will be described for Laplace marginals.

Models for discrete variate time series are described by McKenzie (1985a).

FIRST-ORDER AUTOREGRESSIVE PROCESSES

A natural starting point for nonGaussian modelling of time series is the linear (constant coefficient, additive) first-order autoregressive process given by

$$X_n = \rho X_{n-1} + E_n, \qquad n=0,\pm1,\pm, \ldots; \tag{1}$$

this is a first-order stochastic difference equation (Vervaat, 1979). The E_n's are assumed to be independent and identically distributed (i.i.d.). It will be seen later that it is perhaps a red herring to start with this equation, since linearity and additivity are so closely tied up with Gaussianity. However, one can ask whether i.i.d. innovation sequences $\{E_n\}$ exist such that the X_n's have specified nonGaussian distributions. Before

considering this in the Exponential case, it is necessary to consider and clarify three commonly occurring concepts, namely autoregression, linearity and the Markov property.

(1) Autoregression, Linearity and the Markovian Property.

(i) Autoregression.

In equation (1) it is clear that X_n is explicitly autoregressed (in a linear, additive way) on X_{n-1}, but autoregression is used more broadly in time series analysis and is an important idea in the models given below. Thus, Lawrance and Lewis (1985b) give the three following nested definitions for pth order autoregression:

(a) A stationary sequence $\{X_n\}$ is said to be explicitly (pth order) autoregressive in a linear additive way if the X_n's satisfy the equation

$$X_n = \alpha_1 X_{n-1} + \alpha_2 X_{n-1} + \cdots + \alpha_p X_{n-p} + E_n, \qquad (2)$$

where the E_n are i.i.d. and $\alpha_1, \alpha_2, \cdots, \alpha_p$ are **fixed** parameters. Equation (2) is the standard linear autoregressive model, AR(p).

(b) A more general definition of autoregression of order p could be the **linear conditional expectation** requirement that

$$E(X_n-\mu | X_{n-1}-\mu, X_{n-2}-\mu, \cdots, X_p-\mu) =$$

$$\alpha_1(X_{n-1}-\mu) + \alpha_2(X_{n-2}-\mu) + \cdots + \alpha_p(X_{n-p}-\mu), \qquad (3)$$

where μ is the (stationary) mean of the process. Clearly (2) implies (3), but not vice-versa.

(c) A further and weaker definition of pth-order autoregression is the requirement that the autocorrelations $\{\rho_r\}$ of the process $\{X_n\}$ satisfy the Yule-Walker linear difference equations of order p,

$$\rho_r = \alpha_1\rho_{r-1} + \alpha_2\rho_{r-2} + \cdots + \alpha_p\rho_{r-p}, \qquad r = 1,2,\ldots, \tag{4}$$

for suitable constants $\alpha_1, \alpha_2, \cdots, \alpha_p$ and with $\rho_r = \rho_{-r}$.

Examples of processes which satisfy these definitions will be given below; in particular, it will be seen that most of the models introduced are **linear, additive random coefficient** models which satisfy (3) and/or (4), but not (2).

(ii) Linearity.

To say that a model is linear will be taken to mean that the process $\{X_n\}$ satisfies the equation (2). Random coefficient models, therefore, are nonlinear, although some authors contend that they are linear and that nonlinearity refers only to the case where powers or products of the X_{n-1}, $X_{n-2}, \cdots, E_n$ appear in the defining equation. It will be seen later that there are models which satisfy the third form of autoregression, but are nonlinear by anyone's definition in that they involve products and have parameters which appear as powers.

One should note here, too, that the common practice of transforming time series data, so that it has (approximately) a Gaussian marginal distribution does not imply that the transformed process will have linear structure. It is possible to construct models which have Gaussian marginal distributions, but which are nonlinear.

(iii) Markovian Property.

A third property of interest is the Markovian property, which we will only consider in the first-order case. Thus, a process $\{X_n\}$ is said to (first-order) Markovian if the conditional density of X_n, given the past of the process, satisfies

$$f_{X_n}(x_n|X_{n-1},X_{n-2},\cdots) = f_{X_n}(x_n|X_{n-1}), \tag{5}$$

i.e., given the value of X_{n-1}, the distribution of X_n does not depend on X_{n-2}, X_{n-3},$\cdots$. The process defined by (1) is clearly of this form, by construction, and there are two important consequences of this property. The first is that one can write down the joint density of $X_n,\cdots, X_1$ once we know the marginal density and the conditional density in (5). Thus, a likelihood function can be derived from which one can, in principle, estimate parameters. The second consequence is that one can, for a given marginal distribution, construct a first-order Markovian process from **any** bivariate distribution having the given marginal distribution for its two marginal distributions (Linhart, 1970). This process may not be autoregressive in any of the above senses, although it will possibly have the Yule-Walker autoregression. One objection to this procedure is that there is no guidance as to which of the infinity of, say, bivariate Gamma distributions to use in the construction. The other objection is that without an explicit simple autoregressive structure, it is not only hard to see what is 'going on' in the process, but it is also difficult to extend the process to, for example, moving average structures and simple explicit multivariate processes.

(2) First-Order Processes with Exponential Marginals.

In Equation (1), note that E_n and X_{n-1} are, by definition, independent. Thus the Laplace transform of the distribution of X_n, $\phi_{X_n}(s)$, equals the

Laplace transform of the distribution of X_{n-1} times that of E_n. If the process is stationary, solving this equation gives the equation

$$\phi_E(s) = \phi_X(s)/\phi_X(\rho s). \tag{6}$$

In (6), it is assumed that a solution exists for a given marginal distribution of X. Gaver and Lewis (1980) showed that for X Exponential(λ), E is zero with probability ρ and Exponential(λ) with probability $1-\rho$. Note that this EAR(1) process - Exponential autoregressive of order one - **is a linear process.** What is different is that the innovation random variable, E_n, is not absolutely continuous, since its distribution has a jump at zero. Thus, Mallows (1967) result that linear processes must have Gaussian marginal distributions as ρ approaches one does not hold. However, the process is autoregressive in all three senses. The last sense follows, because for the EAR(1) model $\rho(r)=\rho^r$ for $r=1,2,\ldots$, which is the solution of the first-order Yule-Walker equation. The process may also be written as a random coefficient model

$$X_n = \rho X_{n-1} + I_n Z_n, \tag{7}$$

where the Z_n's are i.i.d. Exponentials and the i.i.d. indicator random variables I_n have $P\{I_n=0\} = 1-P\{I_n=1\} = \rho$.

A problem with the EAR(1) model is that the innovation random variable takes on the value zero with positive probability. This makes the process 'defective' in the sense that when $I_n=0$, then $X_n=\rho X_{n-1}$ and the parameter ρ can be estimated exactly in long enough series (Gaver and Lewis (1980)). Note, too, that sample paths have 'runs down'. Although this bothers people who are accustomed to more conventional looking time series, it is not necessarily bad in hydrological contexts, i.e. river run-offs.

Now note that in (7), we can switch X_{n-1} and Z_n to get another Exponential autoregressive process of order one (in the sense of (1.i.b) and (1.i.c)), which is not defective, exhibits 'runs-up', and is regenerative. The switching which produced this TEAR(1) process led Lawrance and Lewis (1981a) to define the broad, two parameter Exponential autoregressive (in the sense of (1.i.b) and (1.i.c)) process NEAR(1)

$$X_n = K_n X_{n-1} + I_n Z_n, \tag{8}$$

where $0 \leq \beta \leq 1$, $P\{K_n=\beta\} = 1-P\{K_n=0\} = \alpha$ and $P\{I_N=1\} = 1-P\{I_n=(1-\alpha)\beta\} = \delta$, with $\delta = (1-\beta)/\{1-(1-\alpha)\beta\}$. Also, the i.i.d. Exponential sequence $\{Z_n\}$, the i.i.d. $\{K_n\}$ sequence and the i.i.d. $\{I_n\}$ sequence are independent of each other, and K_n and I_n are independent of the X_n's for n-1, n-2,$\cdots$.

This NEAR(1) process includes the EAR(1) process ($\alpha=1$) and the TEAR(1) process ($\beta=1$) given at (7), and can exhibit a wide range of sample path behavior; also $\rho(r) = (\alpha\beta)^{|r|}$. The second form of autoregression follows from the result that $E(X_n|X_{n-1}=x) = \alpha\beta x+(1-\alpha\beta)$, where we assume that the parameter of the Exponential distribution, λ, equals one.

Two other important Exponential processes which will appear as special cases of Gamma models in the next section are the Product Autoregression model of McKenzie (1982) and the Beta-Gamma process of Lewis (1981). We give, now, several other models which are derived from special properties of the Exponential distribution.

Perhaps the most important of the Exponential processes beyond the NEAR(1) process is the Tavares (1977, 1980a, 1980b) process. Like most of the simple models, it exploits probabilistic structures which combine two independent Exponential random variables into a third Exponential random variable. Here the combination is that the minimum of two independent Exponential random variables is Exponential. Thus, if as before, $\{Z_n\}$

Lewis (1981) extended (13) to the very simple bivariate Gamma process $\{X_n', X_n''\}$ where, for $n = 0, \pm 1, \pm 2, \ldots$,

$$X_n' = A_n'(k-q', q')X_{n-1}' + B_n'(q', k-q')G_n(k, \beta), \tag{27}$$

$$X_n'' = A_n''(k-q'', q'')X_{n-1}'' + B_n''(q'', k-q'')G_n(k, \beta). \tag{28}$$

In these equations, $\{A_n', A_n''\}$ and $\{B_n', B_n''\}$ are mutually independent i.i.d. sequences of bivariate Beta variables which, for now, are assumed to be independent pairs. This model may be thought of as giving responses at two different locations with the responses being driven by a common driving effect represented by the common error term $E_n = G_n(k, \beta)$. The random coefficients represent local modifying effects and the autoregressions are local dependencies. A problem with this (too) simple model is that there are only two free parameters, q'and q", for the dependency structure, so that the autocorrelation fixes the crosscorrelation.

There are many ways to extend this model. First, instead of a common error term, $G_n(k, \beta)$, one could have a pair of dependent error terms $G_n'(k, \beta)$ and $G_n''(k, \beta)$ which are a bivariate Gamma pair. Also, one might use different lags in the error terms to represent physical delays. Again, it is possible to cross-couple the processes by switching the lagged terms X_{n-1}' and X_{n-1}'' in (27) and (28) and to make the random coefficients dependent. Gaver and Lewis (1980) used this cross-coupling and negatively correlated random error terms to obtain negative correlation in the Gamma marginal processes.

For further details and other models, see Lewis and Shedler (1979), Jacobs (1978, 1980) and Dewald and Lewis (1985b).

If we replace X_{n-1} in (13) by G_{n-1}, we have a moving average Gamma model whose correlations are zero, except at lag one and are bounded by zero and one-quarter. But, it can be shown (Hugus, 1982) that the attainable range for $\rho(1)$ in random coefficient moving average processes is minus one half to plus one half, as with the normal theory linear MA(1) process. A different structure achieves the maximum range of positive correlations for the Beta-Gamma first-order moving average model (Lewis, McKenzie and Hugus, 1985):

$$X_n(\beta,k) = G_n\{\beta,k/(1+\phi)\} + B_n\{k\phi/(1+\phi),k(1-\phi)/(1+\phi)\}G_{n-1}\{\beta,k/(1+\phi)\}. \quad (26)$$

The lag-one serial correlation is $\rho(1)=\phi/(1+\phi)$, which has its maximum value of one half when $\phi=1/2$. For $\phi=0$ or 1, the process is a sequence of i.i.d. Gamma(β,k) variables. Surprisingly enough, the joint Laplace transform for X_n and X_{n-1} is given by (13), so that the Beta-Gamma autoregressive and moving average processes are very much analogous to the linear Gaussian processes. Like the Gaussian process, these Gamma moving average and autoregressive processes are time reversible.

Because moving average and mixed moving average processes are not Markovian, the problem of parameter estimation in these processes in the nonGaussian case is difficult. Nevertheless, their structures may be realistic for modelling some physical phenomena. The model (26) can be extended to qth-order moving average structures (Lewis, McKenzie and Hugus, 1985).

MULTIVARIATE MODELS

There is a clear need in hydrology and other physical fields for multivariate processes and some such extensions of the simple models we have discussed in previous sections have been made. By way of example of the results which can be obtained as extensions of the simple random coefficient models discussed in the first part of this paper, consider again the Beta-Gamma process.

slightly more complicated for Uniform(0,1) variates; one substitutes 1-U for U.

In other cases of non-symmetric marginal distributions, one solution is to use the antithetic variate X' in place of X in the defining equations, where

$$X' = F_X^{-1}(1 - F_X(X)) \tag{25}$$

and for the symmetric (about zero) case X'=-X. Also, in the Uniform(0,1) case U'=1-U. As an example, consider the Exponential case and substitute for X_{n-2} in the defining equation (20) for the NEAR(2) process the antithetic of X_{n-1}, namely $X'_{n-1} = -\log\{1-\exp(-X_{n-1})\}$. Then one has a process which can exhibit the whole range of correlations for Exponential variables. However, even though the process is explicitly 'autoregressive', it is not autoregressive in any of the three senses given above. It is, however, Markovian, but is probably not simple. Even the correlations, $\rho(r)$, are difficult to calculate.

An alternative scheme for obtaining negative correlation in time series was given in Gaver and Lewis (1980). This consists of cross-coupling two processes. Details are not given here. In the symmetric case it reduces to the usual process. The process is, however, not in general Markovian.

MIXED AUTOREGRESSIVE-MOVING AVERAGE MODELS

Several authors have extended the above autoregressive models to moving average and mixed autoregressive-moving average models for various marginal distributions. We refer the reader to Lawrance and Lewis (1977), Jacobs and Lewis (1977), Lawrance and Lewis (1980), McKenzie (1981) and Lewis, McKenzie and Hugus (1985). It is particularly easy to extend random coefficient models to moving average structures and we discuss here, by way of illustration, only the Beta-Gamma model given at (13).

also has short-term (first-order) dependence. Out of the context of Exponential marginals, the random indexing scheme can be used with any first-order autoregressive scheme to obtain a similar effect. In a somewhat different but related vein, Fernandez and Salas (1985) have extended the autoregressive Gamma process to the case of periodic parameters; this is an extension to the nonGaussian case of the so-called "Thomas-Fiering model", although the model is actually due to Hannan (1955).

We note, too, that it is quite simple to include deterministic periodic and seasonal trends in the models for positive random variables. In particular, the mean in the Gamma random variable is multiplicative and can easily be replaced by a time-varying mean. Log-linear models for this mean are appropriate, since they retain the positivity. In particular, Hugus (1982) fitted an exponential sine to fifteen years of three-hourly wind amplitude readings at ship PAPA in the Gulf of Alaska. The exponential sine included six-hourly, six-monthly and yearly terms. The residual process was fitted by a Beta-Gamma process with $\rho(1)=0.88$.

NEGATIVE CORRELATION

It would be useful to have time series models for nonGaussian data which can accomodate the full range of attainable correlations, although we note that the usual range of $-1 \leq \rho \leq 1$ for Gaussian models is not attainable (Moran, 1967) for positive-valued variables. Thus, for exponential random variables, the lowest value is -0.6449. Attainment of negative correlation in processes in a simple way is intimately tied up with symmetry of the marginal distribution. This is because, when the random variable X is centered at zero, then -X has, by the definition of symmetry, the same distribution as X. Thus, in the extension of the NEAR(2) models to Laplace variables (Dewald and Lewis, 1985a), the whole range of correlations, $-1 < \rho < 1$, is attained by substituting -X for X. The situation is only

$$X_n = \beta_1 K_n' X_{n-1} + \beta_2 K_n'' X_{n-2} + L_n E_n, \qquad n=0,\pm1,\pm2,\cdots, \tag{21}$$

where the i.i.d. sequences $\{L_n\}$ and $\{K_n', K_n''\}$ are assumed to be mutually independent and independent of the i.i.d. Exponential sequence $\{E_n\}$ and of $X_{n-1}, X_{n-2}, \cdots$, $0 \le \beta_1, \beta_2 \le 1$, and $\alpha_1 \ge 0$, $\alpha_2 \ge 0$, $\alpha_1 + \alpha_2 \le 1$ with

$$L_n = \begin{cases} 1 & \text{w.p. } 1 - p_2 - p_3 \\ b_2 & \text{w.p. } p_2 \\ b_3 & \text{w.p. } p_3 \end{cases}, \qquad n = 0,\pm1,\pm2,\cdots, \tag{22}$$

$$(K_n', K_n'') = \begin{cases} (1,0) & \text{w.p. } \alpha_1 \\ (0,1) & \text{w.p. } \alpha_2 \\ (0,0) & \text{w.p. } 1 - \alpha_1 - \alpha_2 \end{cases}, \qquad n = 0,\pm1,\pm2,\cdots. \tag{23}$$

The parameters p_1, p_2, b_1 and b_2 are complicated functions of the four parameters of the process and are not given here. For $\alpha_2=0$, we get the NEAR(1) process and there are several cases which give independent X_n's. Moreover, the correlations, which are positive-valued, satisfy the Yule-Walker equations and we have explicitly

$$\rho(1) = (\alpha_1\beta_1)/(1 - \alpha_2\beta_2) \quad \text{and} \quad \rho(2) = (\alpha_1\beta_1)\rho(1) + (\alpha_2\beta_2). \tag{24}$$

It is fairly certain that this structure can be extended to higher-order autoregressions, but it is not at all clear that explicit expressions for the parameters can be obtained. However, using the mixture scheme given above, this four-parameter second-order process can be extended to higher-orders quite simply (Lawrance and Lewis, 1985a). One can obtain, for instance, a four-parameter 4th-order autoregressive process and this is probably sufficient for most purposes.

(3) Seasonal and Stuttered Models.

The autoregression on X_{n-2} in (21) can be replaced by autoregression on, say, X_{n-s} to give a stationary (non-deterministic) seasonal model, which

$$X_n = A_n X_{n-S_n} + B_n Z_n, \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{20}$$

The assumption is that the process is stationary; the scheme works because a mixture of random variables with identically distributed marginal distributions has that same marginal distribution, **even if the random variables are dependent.** Moreover, it is easily verified that the correlations $\rho(r)$ satisfy the Yule-Walker equation (4) with $\alpha_j = \tau_j E(A_n)$, $j = 1, \cdots, p$.

Drawbacks to this scheme, as to any scheme with this breadth, can be seen to occur for two reasons. One is that the α_j's in the Yule-Walker equations are limited in range, since the τ_j's are probabilities and the $E(A_n)$ will have a limited range in specific cases. Thus, the range of attainable correlations will be limited in comparison with, say, the attainable range of correlations for the AR(p) process. The second drawback comes from the fact that runs of identical values are possible in the sample path of the process, since there is a small probability that, for example, X_n, X_{n+1}, and X_{n+2} may have the value same value X_{n-1} if B_n can take on the value zero. Nevertheless, this random indexing scheme does supply a limited solution to a very difficult and important problem.

(2) The Exponential NEAR(2) MODEL.

A broad extension of the NEAR(1) to second-order autoregression (in the sense of (1.i.b) and 1.i.c)) has been given by Lawrance (1980) and Lawrance and Lewis (1985a). This process is denoted NEAR(2) and has four parameters. Also, despite its restriction to an Exponential marginal distribution, the structure which gives the process is important in extending Exponential processes in general.

Thus, let the stationary process $\{X_n\}$ be defined by the random coefficient stochastic difference equation

Note that the resulting process is not simple and that it is unique. This contrasts to the fact which we have seen above, that there exist many processes with the same marginal distributions and that this may be desirable in the modelling of physical phenomena such as those that occur in hydrology.

HIGHER-ORDER AUTOREGRESSIVE PROCESSES

In many applications there is a need for higher-order nonGaussian autoregressive processes which mimic the AR(p) process given at (2). However, the analog of the simple equation (6) is not available for p>1. In fact, no direct solutions exist to equation (2), other than for Gaussian processes. We describe now several schemes which have been derived, starting with a very general mixture scheme and then describing the NEAR(2) process of Lawrance and Lewis (1985a).

(1) Random Indexing (mixture models).

We note that the first-order autoregressive models given above can be written for the most part as random coefficient models of the form

$$X_n = A_n X_{n-1} + B_n Z_n, \qquad n = 0, \pm 1, \pm 2, \cdots, \tag{19}$$

with the usual independence conditions on the random coefficient sequences A_n and B_n and the innovation sequence Z_n. Note, however, that for given n, the A_n and the B_n do not have to be independent. This is the case for the Beta model given at (17) above. Now the random indexing scheme (Jacobs and Lewis, 1983) is to replace the index n-1 in (19) by $n-S_n$, where S_n is a discrete valued random variable on $1,\ldots,p$ with probabilities $\tau_1, \cdots, \tau_p$ and the S_n's are assumed to be i.i.d. Thus, we have

The Beta-Beta transform states that the product of two independent Beta random variables is a Beta random variable, so that Beta(α,β) $\cdot$ Beta($\alpha+\beta,\gamma$) = Beta($\alpha,\beta+\gamma$). Then the PBAR(1) model is given by

$$X_n = 1 - A_n(1 - B_n X_{n-1}) \qquad n = 1,2,3,\cdots, \tag{17}$$

where $\{A_n\}$ and $\{B_n\}$ are independent sequences of i.i.d. Beta($\beta,\alpha-p$) and Beta($p,\alpha-p$) random variables. The single structural parameter in this scheme, p, determines the correlation structure of the process, which is

$$\rho_X(r) = \{p\beta/\alpha(\alpha+\beta-p)\}^r = \rho^r, \qquad r = 0,1,2,\cdots;\ 0 \le p < \alpha. \tag{18}$$

Note that in this Beta process, the parameters α and β determine the marginal distribution of the X_n's, and p can be freely chosen in the range $0 \le p < \alpha$ to determine the correlation.

An important special case occurs when $\alpha=\beta=1$ and the marginal distribution is Uniform(0,1). This process is an **additive** random coefficient process in Uniform(0,1) variables. **Multiplicative** Uniform(0,1) processes may be obtained, for instance, by negative exponentiation of the NEAR(1) process given at (8). The properties of the resulting two-parameter process have not yet been investigated.

One should note here that the random coefficient approach is not the only way to generate a process with both a specified correlation structure and marginal distribution. There is another approach in the Engineering literature which is to start with a Gaussian ARMA(p,q) process, filter it, transform marginally using a probability integral transform to a Uniform(0,1) marginal and then, via an inverse probability integral transform, create the desired marginal distribution. The filter is chosen to give, if possible, the desired correlation structure. Solutions are known, for example, for an Exponential marginal distribution (Sondhi, 1983).

Another distribution which may be useful for modelling phenomena such as North-South wind velocity components which, unlike the phenomena we have considered before, are not inherently positive valued, is the symmetric ℓ-Laplace distribution. Its characteristic function is $\phi_X(\omega) = (1 + \omega^2)^{-\ell}$, which for $\ell=1$ is that of the Laplace (or double-Exponential) distribution. For ℓ large, this is approximately a Gaussian distribution. For ℓ small, it is peaked at $x=0$ and when $\ell \leq 0.5$, the density is unbounded at $x = 0$. For the Laplace distribution, the NEAR(1) structure (and the NEAR(2) structure given below) go through with analagous formulas to those for the Exponential case (Dewald and Lewis, 1985a) and there is a square-root Beta ℓ-Laplace process which is the analog of the Beta-Gamma process (Dewald, Lewis and McKenzie, 1985).

The Weibull distribution, as well as other extreme-value distributions, is widely used to broaden the Exponential assumption for the marginal distributions of stationary time series. Note that it is not infinitely divisible and, therefore, is not type-L, so that there is no solution to the equation (1). However, a Weibull-distributed random variable is a power-law transform of an Exponentially distributed random variable, so that the minimum structure (9) and the PAR(1) structure (12) give relatively simple first-order autoregressive processes.

Situations do occur in which processes are required with marginal distributions which are bounded below and above. One might, for instance, be interested in modelling a sequence of probabilities, say the probability of overflow at a dam on successive days. For this case, the Beta distribution mentioned above provides a broad, two-parameter distribution model which is widely used. McKenzie (1985b) has derived an autoregressive process for Beta random variables using the Beta-Beta transform. Thus, let Beta(α,β) denote a Beta random variable with probability density function

$$f(x) = \frac{x^{\alpha-1}(1 - x)^{\beta-1}}{B(\alpha,\beta)}, \qquad 0 < x < 1;\ \alpha,\beta > 0. \tag{16}$$

restricted ranges of the parameter ρ. Thus, the mixed Exponential (or hyper-Exponential) distribution, with probability density function

$$f_X(x) = \pi_1\lambda_1 e^{-\lambda_1 x} + \pi_2\lambda_2 e^{-\lambda_2 x}, \quad 0 \leq \pi_1,\pi_2 \leq 1;\ \lambda_1,\lambda_2 \geq 0;\ \pi_1+\pi_2=1, \qquad (15)$$

which is used to model phenomena which are more skewed than the Exponential distribution, is not type-L. However, Gaver and Lewis (1980) gave some sufficient conditions for a solution to exist and Lawrance (1980) gave necessary and sufficient conditions on the parameters β, k, ρ for the solution to exist. The utility of these results are not clear, since the Beta-Gamma process provides an alternative for skewed data which has, in particular, a much simpler estimation solution. There are, however, differences between the two distributions which might be important in modelling.

(5) Ad-hoc Methods for First-order Processes.

Several of the more useful marginal distributions which have been used for modelling marginal distributions in the physical sciences have associated first-order processes. These processes are generally multiplicative or additive random coefficient models, which again illustrates that the linear model (1) may be a false starting point. The processes are generally autoregressive in one of the senses given above. We consider these processes briefly here for Laplace, ℓ-Laplace, mixed Exponential, Weibull and extreme-value distributions.

Lawrance and Lewis (1982) have shown that the switched structure given at (7) has a solution when the X_n process is required to have the mixed Exponential distribution whose density is given at (15). This includes the zero jump Exponential distribution ($1/\lambda_1=0$), which might be useful for modelling winds in areas where there is a positive probability of having an amplitude of zero, i.e., being in the Doldrums.

This Beta-Gamma process is a remarkably 'smooth' process in at least two ways. First, the conditional density of X_n given X_{n-1} is continuous and, therefore, it is relatively simple to get maximum likelihood estimates for the parameters μ, k and q (Hugus, 1982). Secondly, let $\rho=(1-q/k)$. Then the joint Laplace transform of X_n and X_{n-1} is given by (Lewis, McKenzie and Hugus, 1985)

$$\phi(s_1,s_2) = \{(1 + s_1)(1 + s_2)\}^{-k(1-\rho)}(1 + s_1 + s_2)^{-k\rho}. \qquad (14)$$

This result shows, first of all, that the process is time reversible, since s_1 and s_2 can be interchanged in the equation, and secondly, that both the regression of X_n on X_{n-1} and the regression of X_{n-1} on X_n are linear. In these respects the Beta Gamma process is much like the Gaussian AR(1) process. However, in modelling physical phenomena such as river-flows, this smoothness may be a handicap. Thus, it is not possible to model 'runs-down' in data, since for fixed k and μ (marginal distribution), the structural parameter q (or ρ) changes the correlation, but not the sample path behavior. An extension of the process to a broader structure is given in Lawrance and Lewis (1981); it combines the GAR(1) process and the BGAR(1) process.

(4) Other Linear Models.

Other solutions to the first-order, linear, stochastic difference equation (1), known as type-L distributions, are known (Shanbag and Sreehari, 1977; Thorin, 1977a, 1977b), but generally do not give simple, tractable solutions. A case in point is the log-Gamma distribution. Results from McKenzie (1982) for the PAR(1) process say that there is an innovation random variable in this case, but it is not easy to even generate it, except when the X_n's have a log-Exponential distribution. Another problem with use of the linear equation (1) is that useful results may be obtained for

the sense (1.i.a) or the sense (1.i.b) above; however, since $\rho(r)=\rho^r$, $r=1,2,\cdots$, it is autoregressive in the third sense. It is, of course, very nonlinear, in the usual sense of that term.

The last Gamma process we introduce is the Beta-Gamma process (Lewis, 1981) which has been detailed in Lewis, McKenzie and Hugus (1985). This is a simple and very broad process which, in some ways, is an analog of the Gaussian AR(1) process. It is based on the Beta-Gamma transformation (Lewis, 1983) which is as follows. Let X be a Gamma($m+n,\beta$) variate and let Beta(m,n) be an independent Beta variate with parameters $m\geq 0$ and $n\geq 0$; then Beta(m,n)$\cdot$X is a Gamma(m,β) variate. Using this result, we can construct the Beta-Gamma first-order autoregressive process, BGAR(1), as follows:
Let

A_n, $n=1,2,\cdots$, be i.i.d. Beta($k-q,q$) random variables with $q<k$;
B_n, $n=1,2,\cdots$, be i.i.d. Beta($q,k-q$) random variables, independent of the A_n's;
G_n, $n=0,1,\cdots$, be an i.i.d. sequence of Gamma(k,β) **innovation** random variables and let $X_0=G_0$; the G_n sequence is independent of the A_n and the B_n sequence.

Then

$$X_n = A_n X_{n-1} + B_n G_n, \qquad n=1,2,\cdots, \tag{13}$$

is a stationary sequence of Gamma(k,β) variates with $\rho(r)=(1-q/k)^r$, $r=1,2,\cdots$. The process is a random coefficient first-order autoregressive process, and as such, is autoregressive in the second two senses given above. The process is also Markovian. Note that in (13) the term $B_n G_n$ can be written as a Gamma($k-q,\beta$) variate; the form (13) is convenient because it expresses the process, $\{X_n\}$, as an autoregression on i.i.d. Gamma(k,β) variates.

from small positive values to large values, the random variable changes from a highly skewed variable whose density is infinite at zero, to the Exponential (k=1), to a Gaussian (k>50). The density is given by

$$f_X(x) = \beta^k x^{k-1} e^{-x\beta}/\Gamma(k) \qquad \beta>0;\ k>0\ ;\ x\geq 0. \tag{10}$$

Note that the mean, $\mu = k/\beta$, is multiplicative and the Gamma random variable, denoted by $G(\beta,k)$ or $G(\mu,k)$ may be written as, $\mu G(k,k)$ or $\mu G(1,k)$, respectively. The associated Laplace transform is $\phi_X(s)=\{\beta/(\beta+s)\}^k$ and (6) yields, for the linear autoregressive process (1),

$$\phi_E(s) = \{\rho + (1-\rho)\beta/(\beta+s)\}^k, \qquad k>0;\ \ \beta>0, \tag{11}$$

which (Gaver and Lewis, 1980) is the Laplace transform of a proper random variable. The linear, first-order autoregressive process (1) with this innovation random variable is called the Gamma autoregressive process of order one, GAR(1). This process actually was used in hydrology (Bernier, 1970) many years before its rediscovery; unfortunately, although it includes and broadens the EAR(1) process, it too is defective. Thus, E is zero with probability ρ^k, which is very serious if k<1 (very skewed marginal distribution). Also $\phi_E(s)$ cannot be explicitly inverted unless k is an integer, although Lawrance (1982) and McKenzie (1985c) have found methods for generating random variables with this Laplace transform.

McKenzie (1982) introduced a product autoregression process of order one, called PAR(1), of the form

$$X_n = X_{n-1}^{\rho} E_n, \qquad 0 \leq \rho < 1, \tag{12}$$

but, although E_n exists for all values of the Gamma parameter k, it is only possible to generate it for the Exponential case (k=1). Note that this process, since it is multiplicative in structure, is not autoregressive in

denotes a sequence of i.i.d. Exponential(λ) random variables, the minimum process

$$X_n = \min(X_{n-1}/\rho, Z_n), \tag{9}$$

will have Exponential marginals and correlation structure $\rho(r)=\rho^r$, $r=0,1,\cdots$. Thus, the process is autoregressive in the third sense given above, but the autoregression of X_n on X_{n-1}, while explicit, is not linear, even in a random coefficient sense. Nor is the conditional expectation of X_n, given X_{n-1}, linear. Thus, the first two forms of autoregression do not hold. What is remarkable about this process is that it is a **time reversed version** of the EAR(1) process given at (1). This result is due to Chernick, Daley and Littlejohn (1983). In particular it shows that the process is defective. DeHeuvels (1983,1984) has exploited this minimum structure even further and has tried to show that all stationary Exponential processes can be represented as infinite moving minima. However, this generalized minimum structure does not admit Exponential processes with negative correlation and is to some extent defective.

Many other first-order processes in Exponential random variables are now known. For instance, one could add two independent Gaussian AR(1) processes to obtain an Exponential process with geometrically decaying correlations. Suprisingly, this process, while not explicitly autoregressive and not having an explicit structure, has been shown to be Markovian. Another interesting process which arises in the context of engineering geology has been given by McCullagh (1983). Again, negative correlation is not possible.

(3) First-Order Processes with Gamma Marginals.

The two-parameter Gamma distribution is a widely used model for the marginal distribution of many physical time series. As the shape parameter k changes

TIME REVERSIBILITY, HIGHER-ORDER MOMENTS AND RESIDUAL ANALYSIS

A recurrent theme in the models which have been described is that of time reversibility and directionality. Gaussian linear models given by (2), with extensions to mixed autoregressive-moving average structure, are time reversible and are the only dependent linear models with this property (Weiss, 1975). This can be a handicap in modelling, say, river-flow data, which generally has quite marked directionality in its sample path behavior. Models like the NEAR(2) process for Exponential marginals can accommodate a wide variety of directionalities, either gross or subtle, in the sample path behavior. On the other hand, the Beta-Gamma process is time reversible unless it is extended by combination with the GAR(1) process.

It is important to note that serial correlations do not, in any way, reflect directionality in time series. This is apparent from the well known formula that $\rho(r)=\rho(-r)$, $r=0,\pm,1,\pm,\cdots$. This suggests using higher-order moments or their transforms, e.g. bispectra (see for example, Rosenblatt, 1980, 1982).

A simpler approach has been suggested by Lawrance and Lewis (1985a,1985b) for processes which are autoregressive of order p, in the sense that their correlations satisfy the Yule-Walker equations (4). In that case, the **linear autoregressive residuals**

$$R_n = X_n - \alpha_1 X_{n-1} - \cdots - \alpha_p X_{n-p}, \qquad n = p+1,\cdots, \tag{29}$$

are uncorrelated, although not independent unless the residuals are generated by a linear process. It is this difference between lack of dependence and lack of correlation which is exploited in the residual analysis, which consists of examining the autocorrelations of the R_n^2 process, or crosscorrelations between the R_n and the R_n^2 process. This

analysis actually involves higher-order moments of the $\{X_n\}$ process, but uses standard second-order statistical computations.

An extension (Lawrance and Lewis, 1985c) is to use reversed residuals. Thus, since the correlations for the reversed process are the same as for the original process, even if the process is not time reversible, the reversed residuals

$$RR_n = X_n - \alpha_1 X_{n+1} - \cdots - \alpha_p X_{n+p}, \qquad n = 1, \pm 2, \cdots, \tag{30}$$

will also be uncorrelated. An analysis based on both RR_n and R_n is then possible to determine linearity and time-reversibility in the time series and correctness of the fitted models.

CONCLUSIONS

No attempt has been made to completely survey all the models for continuous variate time series which have been proposed. Instead, we have concentrated on marginally specific time series models which are simple in structure and which would be used much in the spirit in which Gaussian linear models are used in the so-called Box-Jenkins method. For hydrologic models which are physically motivated, the reader is referred to Lawrance and Kottegoda (1977) or Salas et al. (1980).

ACKNOWLEDGEMENTS

The work described in this paper owes much to cooperative efforts with Drs. A.J. Lawrance and E. McKenzie. Major Lee Dewald read and corrected the original manuscript. Support during the writing of the paper was supplied by the Office of Naval Research under Grant No. NR042-469.

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